A note on maximal solutions of nonlinear parabolic equations with absorption

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Maximal solutions of nonlinear parabolic equations with absorption

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Abstract We study the existence and the uniqueness of the solution of the problem (P): $\partial_t u - \Delta u + f(u) = 0$ in $Q := \Omega \times (0,\infty)$, $u = \infty$ on the parabolic boundary $\partial_p Q$ when Ω is a domain in $\mathbb{R}^N$ with a compact boundary and f a continuous increasing function satisfying super linear growth condition. We prove that in most cases, the existence and uniqueness is reduced to the same property for the associated stationary equation in Ω.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega := \Gamma$, $Q_T^\Omega := \Omega \times (0,T)$ $(0 < T \leq \infty)$ and $\partial_p Q = \overline{\Omega} \times 0 \cup \partial\Omega \times (0,T]$. We denote by $\rho_{\partial\Omega}(x)$ the distance from x to $\partial\Omega$ and by $d_P(x,t) = \min\{\rho_{\partial\Omega}(x), t\}$ the product distance from $(x,t) \in Q_\infty^\Omega$ to $\partial_p Q_\infty^\Omega$. If $f \in C(\mathbb{R})$, we say that a function $u \in C^{2,1}(Q_\infty^\Omega)$ solution of

$$u_t - \Delta u + f(u) = 0, \tag{1.1}$$

in Q_∞^Ω is a large solution of (1.1) in Q_∞^Ω if it satisfies

$$\lim_{d_P(x,t)\to 0} u(x,t) = \infty. \tag{1.2}$$

The existence of such a u is associated to the existence of large solutions to the stationary equation

$$-\Delta w + f(w) = 0, \tag{1.3}$$

in Ω, i.e. solutions which satisfy

$$\lim_{\rho_{\partial\Omega}(x)\to 0} w(x) = \infty, \tag{1.4}$$

and solutions of the ODE

$$\phi' + f(\phi) = 0 \quad \text{in } (0,\infty). \tag{1.5}$$

subject to the initial blow-up condition

$$\lim_{t\to 0}\phi(t)=\infty. \tag{1.6}$$

A natural assumption on f is to assume that it is nondecreasing with $f(0)\geq 0$. If $f(a)>0$, a necessary and sufficient condition for the existence of a maximal solution $\overline{w}_\Omega$ to (1.3) is the Keller-Osserman condition,

$$\int_a^\infty \frac{ds}{\sqrt{F(s)}}<\infty, \tag{1.7}$$

where $F(s)=\displaystyle\int_0^s f(\tau)d\tau$. A necessary and sufficient condition for the existence of a solution ϕ of (1.6) with initial blow-up is

$$\int_a^\infty \frac{ds}{f(s)}<\infty. \tag{1.8}$$

Furthermore the unique maximal solution $\overline{\phi}$ is obtained by inversion from the formula

$$\int_{\overline{\phi}(t)}^\infty \frac{ds}{f(s)}=t \quad \forall t>0. \tag{1.9}$$

It is known that, if f is convex, (1.7) implies (1.8). If (1.7) holds and there exists a maximal solution to (1.3), it is not always true that this maximal solution is a large solution. In the case of a general nonlinearity, only sufficient conditions are known, independent of the regularity of $\partial\Omega$. We recall some of them.

If $N\geq 3$ and f satisfies the weak singularity assumption

$$\int_a^\infty s^{-2(N-1)/(N-2)}f(s)ds<\infty \quad \forall a>0. \tag{1.10}$$

If $N=2$ and the exponential order of growth of f defined by

$$a_f^+=\inf\left\{a\geq 0:\int_0^\infty f(s)e^{-as}ds<\infty\right\} \tag{1.11}$$

is finite.

When $f(u)=u^q$ with $q>1$, (1.10) means that $q<N/(N-2)$. When $q\geq N/(N-2)$ the regularity of $\partial\Omega$ plays a crucial role in the existence of large solutions. A necessary and sufficient condition involving a Wiener type test which uses the $C_{2,q'}^{\mathbb{R}^N}$-Bessel capacity has been obtained by probabilistic methods by Dhersin and Le Gall [4] in the case $q=2$ and extended to the general case by Labutin [6].

Uniqueness of the large solution of (1.3) has been obtained under three types of assumptions (see [7], [10] and [11]):

If $\partial\Omega=\partial\overline{\Omega}^c$ and $f(u)=u^q$ with $1<q<N/(N-2)$ or if $N=2$ and $f(u)=e^{au}$.

If $\partial\Omega$ is locally a continuous graph and $f(u)=u^q$ with $q>1$ or $f(u)=e^{au}$.

If $f(u) = u^q$ with $q \geq N/(N-2)$ and $C^{\mathbb{R}^N}_{2,q'}(\partial\Omega \setminus \tilde{\overline{\Omega}}^c) = 0$, where $\tilde{E}$ denotes the closure of a set in the fine topology associated to the Bessel capacity $C^{\mathbb{R}^N}_{2,q'}$.

In this article we extend most of the above mentioned results to the parabolic equation (1.1). We first prove that, *if f is super-additive, i. e.*

$$f(x+y) \geq f(x) + f(y) \quad \forall (x,y) \in \mathbb{R} \times \mathbb{R}, \tag{1.12}$$

and satisfies (1.7) and (1.8), there exists a maximal solution $\overline{u}_{Q^\Omega}$ to (1.1) in Q^Ω, and it satisfies

$$\overline{u}_{Q^\Omega}(x,t) \leq \overline{w}_\Omega(x) + \overline{\phi}(t) \quad \forall (x,t) \in Q^\Omega. \tag{1.13}$$

If we assume also that $\partial\Omega = \partial\overline{\Omega}^c$, there holds

$$\max\{\overline{w}_\Omega(x), \overline{\phi}(t)\} \leq \overline{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q^\Omega. \tag{1.14}$$

Under the assumption $\partial\Omega = \partial\overline{\Omega}^c$, it is possible to consider a decreasing sequence of smooth bounded domains Ω^n such that $\overline{\Omega}^n \subset \Omega^{n-1}$, $\bar{\Omega} = \cap\Omega_n$, and prove that the increasing sequence of large solutions $\overline{u}_{Q^{\Omega^n}}$ of (1.1) in $Q^{\Omega^n} := \Omega^n \times (0,\infty)$, converges to the *exterior maximal solution* $\underline{u}_{Q^\Omega}$ of (1.1) in Q^Ω. If we proceed similarly with the large solutions $\overline{w}_{\Omega^n}$ of (1.3) in Ω^n and denote by $\underline{w}_\Omega$ their limit, then *we prove that*

$$\max\{\underline{w}_\Omega(x), \overline{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q^\Omega. \tag{1.15}$$

The main result of this article is the following

Theorem 1. *Assume Ω is a bounded domain such that $\partial\Omega = \partial\overline{\Omega}^c$, $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7), (1.8) and (1.12). Then, if $\underline{w}_\Omega = \overline{w}_\Omega$, there holds $\underline{u}_{Q^\Omega} = \overline{u}_{Q^\Omega}$.*

Consequently, if (1.3) admits a unique large solution in Ω, the same holds for (1.1) in Q^Ω_∞.

2 The maximal solution

In this section Ω is a bounded domain in $\mathbb{R}^N$ and $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7) and (1.8). We set $k_0 = \inf\{\ell \geq 0 : f(\ell) > 0\}$ and assume also that, for any $m \in \mathbb{R}$ there exists $L = L(m) \in \mathbb{R}_+$ such that

$$\forall (x,y) \in \mathbb{R}^2, x \geq m,\ y \geq m \Longrightarrow f(x+y) \geq f(x) + f(y) - L. \tag{2.1}$$

Theorem 2.1 *Under the previous assumptions there exists a maximal solution $\overline{u}_{Q^\Omega}$ in Q^Ω_∞.*

Proof. Step 1- Approximation and estimates. Let Ω_n be an increasing sequence of smooth domains such that $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\cup\Omega_n = \Omega$. For each of these domains and $(n,k) \in \mathbb{N}^2_*$ we denote by $w = w_{n,k}$ the solutions of

$$\left\{ \begin{array}{rl} -\Delta w + f(w) = 0 & \text{in } \Omega_n \\ w = k & \text{in } \partial\Omega_n. \end{array} \right. \tag{2.2}$$

where $\partial_p Q_\infty^{\Omega_n} := \partial\Omega_n \times (0,\infty) \cup \overline{\Omega}_n \times \{0\}$. By [5] there exists a decreasing function g from $\mathbb{R}_+$ to $\mathbb{R}$, with limit ∞ at zero, such that

$$w_{n,k}(x) \leq g\left(\rho_{\partial\Omega_n}(x)\right) \quad \forall x \in \Omega_n. \tag{2.3}$$

The mapping $k \to w_{n,k}$ is increasing, while $n \to w_{n,k}$ is decreasing. If we set

$$\overline{w}_\Omega = \lim_{n\to\infty} \lim_{k\to\infty} w_{n,k}, \tag{2.4}$$

it is classical that $\overline{w}_\Omega$ is the maximal solution of (1.3) in Ω, and it satisfies

$$w(x) \leq g\left(\rho_{\partial\Omega}(x)\right) \quad \forall x \in \Omega. \tag{2.5}$$

We denote also by $u = u_{n,k}$ the solution of

$$\left\{\begin{array}{rl} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^{\Omega_n} \\ u = k & \text{in } \partial_p Q_\infty^{\Omega_n}. \end{array}\right. \tag{2.6}$$

By the maximum principle $k \to u_{n,k}$ is increasing and $n \to u_{n,k}$ decreasing. If we denote by $\bar\phi$ the maximal solution of the ODE (1.5), then $\bar\phi(t)$ is expressed by inversion by (1.9). If $t_k = \bar\phi^{-1}(k)$, there holds, since $\bar\phi$ is decreasing,

$$\bar\phi(t + t_k) \leq u_{n,k}(x,t) \quad \text{in } Q_\infty^{\Omega_n}. \tag{2.7}$$

Furthermore, if $f(k) \geq 0$ (which holds if $k \geq k_0$), $w_{n,k} \leq k$. Therefore

$$w_{n,k}(x) \leq u_{n,k}(x,t) \quad \text{in } Q_\infty^{\Omega_n}. \tag{2.8}$$

Combining (2.7) and (2.8), we derive

$$\max\{w_{n,k}(x), \bar\phi(t+t_k)\} \leq u_{n,k}(x,t) \quad \forall (x,t) \in Q_\infty^{\Omega_n}. \tag{2.9}$$

Next we obtain an upper estimate. Let $T > 0$ and $m \in \mathbb{R}$ such that

$$\min\{\overline{w}_\Omega(x) : x \in \Omega\} > m \geq \bar\phi(T).$$

For $n \geq n_1$ and $k \geq k_1$ there holds $\min\{w_{n,k}(x) : x \in \Omega\} \geq m$. Let $L = L(m) \geq 0$ be the corresponding damping term from (2.1). If $v_{n,k} = w_{n,k}(x) + \bar\phi(t+t_k)$, then it satisfies

$$v_t - \Delta v + f(v) = f(v) - f(\bar\phi(.+t_k)) - f(w_{n,k}) \geq -L \quad \text{if } (x,t) \in \Omega_n \times [0, T - t_k]. \tag{2.10}$$

Since $L \geq 0$, the function $\tilde v_{n,k} := v_{n,k} + Lt$ is a supersolution for (1.1) in $Q_{T-t_k}^{\Omega_n} := \Omega_n \times (0, T - t_k)$ which dominates $u_{n,k}$ on $\partial_p Q_{T-t_k}^{\Omega_n}$, thus in $Q_{T-t_k}^{\Omega_n}$ by the maximum principle. Therefore

$$u_{n,k}(x,t) \leq w_{n,k}(x) + \bar\phi(t+t_k) + Lt \quad \forall (x,t) \in Q_{T-t_k}^{\Omega_n}. \tag{2.11}$$

Step 2- Final estimates and maximality. Using the different monotonicity properties of the mapping $(k,n) \mapsto w_{n,k}$ and the estimates (2.9) and (2.11), it follows that the function defined by

$$\overline{u}_{Q^\Omega} := \lim_{n\to\infty} \lim_{k\to\infty} u_{n,k} \tag{2.12}$$

is a solution of (1.1) in Q^{Ω}_{∞}. Furthermore

$$\max\{\overline{w}_{\Omega}(x),\bar{\phi}(t)\} \leq \overline{u}_{Q^{\Omega}}(x,t) \quad \forall (x,t) \in Q^{\Omega}_{\infty}, \tag{2.13}$$

and

$$\overline{u}_{Q^{\Omega}}(x,t) \leq \overline{w}_{\Omega}(x) + \bar{\phi}(t) + tL(\phi(T)) \quad \forall (x,t) \in Q^{\Omega}_{T}. \tag{2.14}$$

since $\phi(T) \leq \min\{\overline{w}_{\Omega}(x) : x \in \Omega\}$. Next, we consider $u \in C^{2,1}(Q^{\Omega}_{\infty})$, solution of (1.1) in Q^{Ω}_{∞}. Then, for $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $k^* > 0$ such that for $k \geq k^*$,

$$u_{n,k}(x,t-\epsilon) \geq u(x,t) \quad \forall (x,t) \in \Omega_n \times (\epsilon,\infty).$$

Letting successively $k \to \infty$, $n \to \infty$ and $\epsilon \to 0$, yields to $\overline{u}_{Q^{\Omega}} \geq u$ in Q^{Ω}_{∞}. □

Since $\overline{w}_{\Omega}$ be a large solution in Ω implies the same boundary blow-up for $\overline{u}_{Q^{\Omega}}$ on $\partial\Omega \times (0,\infty)$, we give below some conditions which implies that $\overline{u}_{Q^{\Omega}}$ is a large solution.

Corollary 2.2 *Assume the assumptions of Theorem 2.1 are fulfilled. Then $\overline{u}_{Q^{\Omega}}$ is a large solution if one of the following additional conditions is satisfied:*

(i) $N \geq 3$ and f satisfies the weak singularity condition (1.10).

(ii) $N = 2$ and the exponential order of growth of f defined by (1.11) is positive.

(iii) $N \geq 3$ and $\partial\Omega$ satisfies the Wiener regularity criterion.

Proof. Under condition (i) or (ii), for any $x_0 \in \partial\Omega$, there exists a solution w_{c,x_0} of

$$\left\{ \begin{array}{rl} -\Delta w + f(w) = c\delta_{x_0} & \text{in } B_R(x_0) \\ w = 0 & \text{in } \partial B_R(x_0), \end{array} \right. \tag{2.15}$$

where $R > 0$ is chosen such that $\overline{\Omega} \subset B_R(x_0)$ and $c > 0$ is arbitrary under condition (i) and smaller that $2/a^{+}_{f}$ in case (ii). The function w_{c,x_0} is radial with respect to x_0 and

$$\lim_{x \to x_0} w_{c,x_0}(x) = \infty.$$

If $x \in \Omega$, we denote by x_0 a projection of x on $\partial\Omega$. Since

$$w_n(x) \geq w_{c,x_0}(x) \Longrightarrow \overline{w}_{\Omega}(x) \geq w_{c,x_0}(x),$$

we derive from (2.13),

$$\lim_{\rho_{\partial\Omega}(x) \to 0} \overline{u}_{Q^{\Omega}}(x,t) = \infty,$$

uniformly with respect to $t > 0$. In case (iii) we see that, for any $k > 0$

$$\overline{w}_{\Omega}(x) \geq w_{k,\infty}(x) \quad \forall x \in \Omega, \tag{2.16}$$

where $w_{k,\infty}$ is the solution of (2.2), with Ω_n replaced by Ω. This again implies (2.13). □

Using estimate (2.13) leads to the asymptotic behavior of $\overline{u}_{Q^{\Omega}}(x,t)$ when $t \to \infty$.

Corollary 2.3 *Assume the assumptions of Theorem 2.1 are fulfilled. Then* $\overline{u}_{Q^\Omega}(x,t) \to \overline{w}_\Omega(x)$ *locally uniformly on* Ω *when* $t \to \infty$.

Proof. For any $k > k_0$ and $n \in \mathbb{N}_*$ and any $s > 0$, there holds by the maximum principle,

$$u_{n,k}(x,s) \leq k = u_{n,k}(x,0) \quad \forall x \in \Omega_n.$$

Using the monotonicty of f, we derive $u_{n,k}(x,t+s) \leq u_{n,k}(x,t)$ for any $(x,t) \in Q_\infty^{\Omega_n}$. Letting $k \to \infty$ and then $n \to \infty$ yields to

$$\overline{u}_{Q^\Omega}(x,t+s) \leq \overline{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q_\infty^\Omega. \tag{2.17}$$

It follows that $\overline{u}_{Q^\Omega}(x,t)$ converges to some $W(x)$ as $t \to \infty$ and $\overline{w}_\Omega \leq W$ from (2.13). Using the parabolic equation regularity theory, we derive that the trajectory $\mathcal{T} := \bigcup_{t\geq 0}\{\overline{u}_{Q^\Omega}(.,t)\}$ is compact in the $C^1_{loc}(\Omega)$-topology. Therefore W is a solution of (1.3) in Ω. It coincides with $\overline{w}_\Omega$ because of the maximality. □

3 Large solutions

In this section we construct a minimal-maximal solution of (1.1) which is the minimal large solution whenever it exists. If $\partial\Omega$ is regular enough, the construction of the minimal large solution is easy.

Theorem 3.1 *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *the boundary of which satisfies the Wiener regularity condition. If* $f \in C(\mathbb{R})$ *is nondecreasing and satisfies (1.7), (1.8) and (2.1), then there exists a minimal large solution* $\underline{u}_{Q^\Omega}$ *to (1.1) in* Q_∞^Ω. *Furthermore*

$$\max\{\underline{w}_\Omega(x), \overline{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q_\infty^\Omega, \tag{3.1}$$

and, for any $T > 0$,

$$\underline{u}_{Q^\Omega}(x,t) \leq \underline{w}_\Omega(x) + \overline{\phi}(t) + tL(\overline{\phi}(T)) \quad \forall (x,t) \in Q_T^\Omega, \tag{3.2}$$

where $L(\overline{\phi}(T))$ *is as in (2.16), and* $\underline{w}_\Omega$ *denotes the minimal large solution of (1.3) in* Ω.

Proof. For $k \geq k_0$ (see Section 2), we denote by $\underline{u}_k$ the solution of

$$\left\{\begin{aligned} u_t - \Delta u + f(u) &= 0 \quad \text{in } Q_\infty^\Omega \\ u &= k \quad \text{in } \partial_p Q_\infty^\Omega. \end{aligned}\right. \tag{3.3}$$

When k increases, u_k increases and converges to some large solution $\underline{u}_{Q^\Omega}$ of (1.1) in Q_∞^Ω. If u is any large solution of (1.1) in Q_∞^Ω, then the maximum principle and (1.2) implies $u \geq u_k$. Therefore $u \geq \underline{u}_{Q^\Omega}$. The same assumption allows to construct the solution w_k of

$$\left\{\begin{aligned} -\Delta w + f(w) &= 0 \quad \text{in } \Omega \\ w &= k \quad \text{in } \partial\Omega, \end{aligned}\right. \tag{3.4}$$

and, by letting $k \to \infty$, to obtain the minimal large solution $\underline{w}_\Omega$ of (1.3) in Ω. Next we first observe, that, as in the proof of Theorem 2.1, (2.10) applies under the form

$$\overline{\phi}(t + t_k) \leq u_k(x,t) \quad \text{in } Q^\Omega_\infty, \tag{3.5}$$

where, we recall it, $t_k = \overline{\phi}^{-1}(k)$. In the same way, for $k \geq k_0$ (with $f(k) \geq 0$), (2.11) holds under the form

$$w_k(x) \leq u_k(x,t) \quad \text{in } Q^\Omega_\infty. \tag{3.6}$$

Letting $k \to \infty$ yields to

$$\max\{\underline{w}_\Omega(x), \overline{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q^\Omega_\infty. \tag{3.7}$$

In order to prove the upper estimate we consider the same m as it the proof of Theorem 2.1 such that $\min\{\min\{w_k(x) : x \in \Omega\}, \overline{\phi}(t)\} \geq m$, and for $k' > k$, there holds

$$w_{k'} + \overline{\phi} \geq k = w_k\big|_{\partial_p Q^\Omega_T}.$$

Since $w_{k'}(x) + \overline{\phi}(t) + tL$ is a supersolution for (1.1) in Q^Ω_T it follows $w_{k'} + \overline{\phi} + tL \geq w_k$ in Q^Ω_T. Letting successively $k' \to \infty$ and $k' \to \infty$, we derive (3.2). $\square$

From this result we can deduce uniqueness results for solution of

Corollary 3.2 *Under the assumptions of Theorem 3.1, if we assume moreover that f is convex and, for any $\theta \in (0,1)$, there exists r_θ such that*

$$r \geq r_\theta \Longrightarrow f(\theta r) \leq \theta f(r). \tag{3.8}$$

Then

$$\underline{w}_\Omega = \overline{w}_\Omega \Longrightarrow \underline{u}_{Q^\Omega} = \overline{u}_{Q^\Omega}. \tag{3.9}$$

Proof. We fix $T \in (0,1]$ such that

$$tL(\overline{\phi}(1)) \leq \overline{\phi}(t) \quad \forall t \in (0,T],$$

(remember that L is always positive) and

$$2\underline{w}_\Omega(x) + \overline{\phi}(t) \geq 0 \quad \forall (x,t) \in Q^\Omega_T.$$

Then $\underline{w}_\Omega(x) + \overline{\phi}(t) \geq 0$ and

$$\underline{w}_\Omega(x) + \overline{\phi}(t) + tL(\overline{\phi}(1)) \leq \underline{w}_\Omega(x) + 2\overline{\phi}(t) \leq \underline{w}_\Omega(x) + 2\overline{\phi}(t) \leq 3\left(\underline{w}_\Omega(x) + \overline{\phi}(t)\right),$$

from which inequality follows

$$2^{-1}\left(\underline{w}_\Omega(x) + \overline{\phi}(t)\right) \leq \underline{u}_{Q^\Omega}(x,t) \leq 3\left(\underline{w}_\Omega(x) + \overline{\phi}(t)\right) \quad \forall (x,t) \in Q^\Omega_T.$$

Therefore, if $\underline{w}_\Omega = \overline{w}_\Omega$, it follows

$$\underline{u}_{Q^\Omega} \leq \overline{u}_{Q^\Omega} \leq 6\underline{u}_{Q^\Omega} \quad \text{in } Q^\Omega_T. \tag{3.10}$$

Next we assume $\underline{u}_{Q^\Omega} < \overline{u}_{Q^\Omega}$ and set

$$u^* = \underline{u}_{Q^\Omega} - \frac{1}{6}\left(\overline{u}_{Q^\Omega} - \underline{u}_{Q^\Omega}\right).$$

Since f is convex, u^* is a supersolution of (1.1) in Q_T^Ω (see [8], [10]) and $u^* < \underline{u}_{Q^\Omega}$. Up to take a smaller T, we can also assume from (3.8) that $\min\{\underline{u}_{Q^\Omega}(x,t) : (x,t) \in Q_T^\Omega\} \geq r_{1/12}$, thus

$$f(\underline{u}_{Q^\Omega}/12) \leq \frac{1}{12} f(\underline{u}_{Q^\Omega}) \quad \text{in } Q_T^\Omega.$$

Therefore $\underline{u}_{Q^\Omega}/12$ is a subsolution for (1.1) in Q_T^Ω and $12^{-1}\underline{u}_{Q^\Omega} < u^*$. Using a standard result of sub and super solutions and the fact that f is locally Lipschitz continuous, we see that there exists some $u^\#$ solution of (1.1) in Q_T^Ω such that

$$\frac{1}{12}\underline{u}_{Q^\Omega} \leq u^\# \leq u^* < \underline{u}_{Q^\Omega} \quad \text{in } Q_T^\Omega. \tag{3.11}$$

Then $u^\#$ is a large solution, which contradicts the minimality of $\underline{u}_{Q^\Omega}$ on Q_T^Ω. Finally $\underline{u}_{Q^\Omega} = \overline{u}_{Q^\Omega}$ in Q_∞^Ω. $\square$

Lemma 3.3 *Let Ω be a bounded domain in $\mathbb{R}^N$ and, for $\epsilon > 0$, $\Omega_\epsilon := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \overline{\Omega}) < \epsilon\}$. The four following assertions are equivalent:*
(i) $\partial\Omega = \partial\overline{\Omega}^c$.
(ii) For any $x \in \partial\Omega$, there exists a sequence $\{x_n\} \subset \overline{\Omega}^c$ such that $x_n \to x$.
(iii) For any $x \in \partial\Omega$ and any $\epsilon > 0$, $B_\epsilon(x) \cap \overline{\Omega}^c \neq \emptyset$.
(iv) For any $x \in \partial\Omega$, $\lim_{\epsilon\to 0} \operatorname{dist}(x, \Omega_\epsilon^c) = 0$.
(v) $\Omega = \overset{o}{\overline{\Omega}}$.

Proof. There always holds $\partial\overline{\Omega}^c = \overline{\overline{\Omega}^c} \cap \overline{\Omega} \subset \Omega^c \cap \overline{\Omega} = \partial\Omega$.
(i)$\Longrightarrow$ (iii). Assume (iii) does not hold, there exist $x_0 \in \partial\Omega$ and $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x_0) \cap \overline{\Omega}^c = \emptyset$. Thus $x_0 \notin \overline{\overline{\Omega}^c}$, and $x_0 \notin \partial\overline{\Omega}^c$. Therfore (i) does not hold.
(iii)$\Longrightarrow$ (i). Let $x_0 \in \partial\Omega$. If, for any $\epsilon > 0$, $B_\epsilon(x) \cap \overline{\Omega}^c \neq \emptyset$, then $x \in \overline{\overline{\Omega}^c}$. Because $x \in \Omega^c \cap \overline{\Omega}$, it implies that $x \in \overline{\Omega} \cap \overline{\overline{\Omega}^c} = \partial\overline{\Omega}^c$.
The equivalence between (iii) and (ii) is obvious.
(ii))$\Longrightarrow$ (iv). We assume (iv) does not hold. There exist $x_0 \in \partial\Omega$, $\alpha > 0$ and a sequence of positive real numbers $\{\epsilon_n\}$ converging to 0 such that $\operatorname{dist}(x_0, \Omega_{\epsilon_n}^c) \geq \alpha$. Since for $\epsilon \geq \epsilon_n$, $\Omega_\epsilon^c \subset \Omega_{\epsilon_n}^c$, there holds $\operatorname{dist}(x_0, \Omega_\epsilon^c) \geq \alpha$. Furthermore, this inequality holds for any $\epsilon > 0$. If there exist a sequence $\{x_n\} \subset \overline{\Omega}^c$ such that $x_n \to x_0$, then $\operatorname{dist}(x_n, \overline{\Omega}) = \delta_n > 0$, thus $x_n \in \Omega_{\delta_n}^c$. Consequently $|x_n - x_0| \geq \alpha$, which is impossible. Therefore (ii) does not hold.
(iv)$\Longrightarrow$ (iii). Let $x \in \partial\Omega$ and $x_n \in \Omega_{1/n}^c$ such that $|x - x_n| = \operatorname{dist}(x, \Omega_{1/n}^c) \to 0$. Since $\Omega_{1/n}^c \subset \overline{\Omega}$, $x_n \in \overline{\Omega}^c$ and $x_n \to x$.

(iii)$\Longrightarrow$ (v). We first notice that $\overline{\Omega} = \cap_{\epsilon>0}\Omega_\epsilon = \cap_{\epsilon>0}\overline{\Omega}_\epsilon$ and $\Omega \subset \overset{o}{\overline{\Omega}}$. If there exists some $x \in \overset{o}{\overline{\Omega}} \setminus \Omega$, then for some $\epsilon > 0$, $B_\epsilon(x) \subset \overline{\Omega}$ which implies $B_\epsilon(x) \cap \overline{\Omega}^c = \emptyset$. But $x \notin \Omega$ implies $x \in \partial\Omega$. Thus (iii) does not hold.

(v)$\Longrightarrow$ (iii). If (iii) does not hold, there exists $x \in \partial\Omega$ and $\epsilon > 0$ such that $B_\epsilon(x) \cap \overline{\Omega}^c = \emptyset \Longleftrightarrow B_\epsilon(x) \subset \overline{\Omega}$. Therefore $x \in \overset{o}{\overline{\Omega}} \setminus \Omega$. □

Definition 3.4 *A solution U (resp. W to problem (1.1) in Q_∞^Ω (resp. (1.3) in Ω) is called an exterior maximal solution if it is larger than the restriction to Q_∞^Ω (resp. Ω) of any solution of (1.1) (resp. (1.3)) defined in an open neighborhood of Q_∞^Ω (resp. Ω)).*

Proposition 3.5 *Assume Ω is a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega = \partial\overline{\Omega}^c$ and $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7). Then there exists an exterior maximal solution $\underline{w}_\Omega^*$ to problem (1.3) in Ω.*

Proof. Since $\partial\Omega = \partial\overline{\Omega}^c$ we can consider the decreasing sequence of the $\Omega_{1/n}$ defined in Lemma 3.3 with $\epsilon = 1/n$ and, for each n, the minimal large solutions $\underline{w}_n$ of (1.3) in $\Omega_{1/n}$: this possible since $\partial\Omega_{1/n}$ is Lipschitz. The sequence $\{\underline{w}_n\}$ is increasing. Its restriction to Ω is bounded from above by the maximal solution $\overline{w}_\Omega$. It converges to some function $\underline{w}_\Omega^*$. By Lemma 3.3-(v), $\underline{w}_\Omega^*$ is a solution of (1.3) in the interior of $\cap_n\Omega_{1/n}$ which is Ω. If w is any solution of (1.3) defined in an open neighborhood of $\overline{\Omega}$, it is defined in $\Omega_{1/n}$ for n large enough and therefore smaller than $\underline{w}_n$. Thus $w|_\Omega \leq \underline{w}_\Omega^*$. Consequently, $\underline{w}_\Omega^*$ coincides with the supremum of the restrictions to Ω of solutions of (1.3) defined in an open neighborhood of $\overline{\Omega}$. □

Proposition 3.6 *Let $f \in C(\mathbb{R})$ be a nondecreasing function for which (1.7) holds and Ω a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega = \partial\overline{\Omega}^c$. Then $\underline{w}_\Omega^*$ is smaller than any large solution. Furthermore, if $\partial\Omega$ satisfies the Wiener regularity criterion and is locally the graph of a continuous function, then $\underline{w}_\Omega = \underline{w}_\Omega^*$.*

Proof. We first notice that Wiener criterion implies statement (iii) in Lemma 3.3, hence $\partial\Omega = \partial\overline{\Omega}^c$. If w_Ω is a large solution, it dominates on $\partial\Omega$, and therefore in Ω by the maximum principle, the restriction to Ω of any function w solution of (1.3) in an open neighborhood of $\overline{\Omega}$. Then

$$\underline{w}_\Omega^* \leq w_\Omega.$$

Consequently, if $\underline{w}_\Omega^*$ is a large solution, it coincides with the minimal large solution $\underline{w}_\Omega$. Because $\partial\Omega$ is compact, there exists a finite number of bounded open subset $\mathcal{O}_j$, hyperplanes H_j and continuous functions h_j from $H_j \cap \overline{\mathcal{O}}_j$ into $\mathbb{R}_+$ such that

$$\partial\Omega \cap \overline{\mathcal{O}}_j = \left\{x = x' + h_j(x')\nu_j : \forall x' \in H_j \cap \overline{\mathcal{O}}_j\right\}$$

where ν_j is a fixed unit vector orthogonal to H_j and $\partial\Omega \subset \cup_j\mathcal{O}_j$. We can assume that $H_j \cap \overline{\mathcal{O}}_j = \overline{B}_j$ is a (N-1) dimensional closed ball and,

$$G_j := \{x = x' + t\nu_j : x' \in \overline{B}_j,\, 0 \leq t < h_j(x')\} \subset \Omega,$$

$$G_j^\# := \{x = x' + t\nu_j : x' \in \overline{B}_j,\, h_j(x') < t \leq a\} \subset \overline{\Omega}^c.,$$

for some $a > 0$ such that $a/4 < h_j(x') < 3a/4$ for any $x' \in \overline{B}_j$. Finally, we can assume that

$$\mathcal{O}_j = \{x = x' + t\nu_j : \ x' \in \overline{B}_j, \, 0 \le t \le a\}.$$

Let $\epsilon \in (0, a/8)$ and

$$G_{j,\epsilon} := \{x = x' + t\nu_j : \ x' \in \overline{B}_j, \, \epsilon \le t < h_j(x') + \epsilon\}.$$

There exists a smooth bounded domain Ω' such that $\overline{\Omega} \subset \Omega'$ and

$$\partial\Omega' \cap \overline{\mathcal{O}}_j = \{x = x' + \ell(x')\nu_j : x' \in \overline{B}_j, \, h(x') + \epsilon/2 \le \ell(x') \le h(x') + 3\epsilon/2\},$$

where $\ell \in C^\infty(\overline{B}_j)$. We denote $G_j := G_{j,0}$,

$$\partial_p G_{j,\epsilon} := \{x = x' + t\nu_j : \ x' \in \partial B_j, \, \epsilon \le t \le h_j(x') + \epsilon\} \cup \{x = x' + \epsilon\nu_j : \ x' \in B_j\},$$

and

$$\partial_u G_{j,\epsilon} := \{x = x' + (h_j(x') + \epsilon)\nu_j : \ x' \in B_j\}.$$

Let w' be the minimal large solution of (1.3) in Ω', $\alpha' = \min\{w'(x) : x \in \Omega'\}$ and W_ϵ the minimal solution of

$$\left\{\begin{array}{rl} -\Delta W + f(W) = 0 & \text{in } G_{j,\epsilon} \\ W = \alpha' & \text{in } \partial_p G_{j,\epsilon} \\ \lim\limits_{t \to h(x')+\epsilon} W(x' + t\nu_j) = \infty & \forall x' \in B_j. \end{array}\right. \tag{3.12}$$

Then $w' \ge W_\epsilon$ in $G_{j,\epsilon} \cap \Omega'$. Furthermore $W_\epsilon(x) = W_\epsilon(x' + t\nu_j) = W_0(x' + (t - \epsilon)\nu_j)$ for any $x' \in \overline{B}_j$ and $\epsilon < t < h(x') + \epsilon$. Therefore, given $k > 0$, there exists $\delta_k > 0$ such that for any

$$x' \in \overline{B}_j \text{ and } h_j(x') - \delta_k \le t < h_j(x') \Longrightarrow W_0(x' + t\nu_j) \ge k.$$

As a consequence, $\liminf_{t \to h_j(x')} \underline{w}^*_\Omega(x' + t\nu_j) \ge k$, uniformly with respect to $x' \in \overline{B}_j$. This implies that $\underline{w}^*_\Omega$ is a large solution. □

Remark. We conjecture that the equality $\underline{w}^*_\Omega = \underline{w}_\Omega$ holds under the mere assumption that the Wiener criterion is satisfied.

Theorem 3.7 *Assume Ω is a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega = \partial\overline{\Omega}^c$ and $f \in C(\mathbb{R})$ satisfies (1.7), (1.8) and (2.1). Then there exists a exterior maximal solution $\underline{u}^*_{Q^\Omega}$ to problem (1.1). Furthermore estimates (3.1) and (3.2) hold with $\underline{w}_\Omega$ replaced by the exterior maximal solution $\underline{w}^*_\Omega$ to problem (1.3) in Ω.*

Proof. The construction of $\underline{u}^*_{Q^\Omega}$ is similar to the one of $\underline{w}_\Omega$, since we can restrict to consider open neighborhoods $Q_{1/n} = \Omega_{1/n} \times (-1/n, \infty)$. Then $\underline{u}^*_{Q^\Omega}$ is the increasing limit of the minimal large solutions u_n of (1.1) in $Q_{1/n}$, since $\overline{Q^\Omega_\infty} = \cap_n Q_{1/n}$ and, by Lemma 3.3-(v), $Q^\Omega_\infty = \overset{o}{\overline{Q^\Omega_\infty}}$. We recall that the minimal large solution w_n of (1.3) in $\Omega_{1/n}$ is the increasing limit, when $k \to \infty$, of the sequence of solution $\{w^k_n\}$ of

$$\left\{\begin{array}{rl} -\Delta w + f(w) = 0 & \text{in } \Omega_{1/n} \\ w = k & \text{on } \partial\Omega_{1/n}, \end{array}\right. \tag{3.13}$$

while the minimal large solution u_n of (1.1) in $Q_{1/n}$ is the (always increasing) limit of the solutions u_n^k of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_{1/n} \\ u = k & \text{on } \partial_p Q_{1/n}. \end{cases} \tag{3.14}$$

Clearly

$$\max\{w_n^k, \overline{\phi}(. + 1/n)\} \leq u_n(x,t),$$

which implies (3.1). For the other inequality, we see that $(x,t) \mapsto w_n^k(x) + \overline{\phi}(t) + Lt$ is a supersolution which dominates u_n^k on ∂_p, where L corresponds to the minimum of w_n^k in $\Omega_{1/n} Q_{1/n}$. Thus

$$u_n(x,t) \leq w_n^k + \overline{\phi}(. + 1/n),$$

which implies

$$\max\{\underline{w}_\Omega^*(x), \overline{\phi}(t)\} \leq \underline{u}_\Omega^*(x,t) \quad \forall (x,t) \in Q_\infty^\Omega. \tag{3.15}$$

The upper estimate is proved in the following way. If $k > n$, $\overline{Q}_k \subset Q_n$. Therefore, choosing m such that $\min\left\{\min\{\underline{w}_{\Omega_{1/k}}(x) : x \in \Omega_{1/k}, \min\{\overline{\phi}(t+1/k) : t \in (0,T]\}\right\} \geq m$, we obtain that $(x,t) \mapsto \underline{w}_{\Omega_{1/k}}(x) + \phi(t+1/k) + Lt$ is a super solution of (1.1) in $Q_T^{\Omega_{1/k}}$, thus it dominates the minimal large solution of (1.1) in $Q_T^{\Omega_{1/n}}$. Letting successively $k \to \infty$ and $n \to \infty$, yields to

$$\underline{u}_\Omega^*(x,t) \leq \underline{w}_\Omega^*(x) + \overline{\phi}(t) \quad \forall (x,t) \in Q_T^\Omega. \tag{3.16}$$

□

The next result extends Corollary 3.2 without the boundary Wiener regularity assumption.

Theorem 3.8 *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega = \partial\overline{\Omega}^c$. If $f \in C(\mathbb{R})$ is convex and satisfies (1.7), (1.8), (2.1) and (3.8). Then, if $\underline{w}_\Omega^*$ is a large solution, the following implication holds*

$$\underline{w}_\Omega^* = \overline{w}_\Omega \Longrightarrow \underline{u}_{Q^\Omega}^* = \overline{u}_{Q^\Omega}. \tag{3.17}$$

Proof. If $\underline{w}_\Omega^*$ is a large solution, the same is true for $\underline{u}_{Q^\Omega}^*$ because of (3.1). Actually $\underline{u}_{Q^\Omega}^*$ is the minimal large solution in Q_∞^Ω for the same reasons as $\underline{w}_\Omega^*$. Therefore the proof of Corollary 3.2 applies and it implies the result. □

Remark. We conjecture that (3.17) holds, even if $\underline{w}_\Omega^*$ is not a large solution.

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